

TEMA 2.

Punto interior $\rightarrow x_0$ es punto interior de un cto $A \subset \mathbb{R} \Leftrightarrow \exists$ un entorno

$$(x_0 - r, x_0 + r) \subset A$$

DERIVADA: $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$

Ej. (M1.) $f(x) = x^2$ $f'(x)$ en $x_0 = 3$

$$\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \underset{f(x)=x^2}{=} \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h \rightarrow 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} = \lim_{h \rightarrow 0} \frac{h(6+h)}{h} = \lim_{h \rightarrow 0} (6+h) = 6,$$

RECTA Tg

$$\boxed{y - f(x_0) = f'(x_0)(x - x_0)}$$
 recta tg a $f(x)$ en $x = x_0$

cont. M1: $f(x) = x^2$ $x_0 = 3$

$$f'(x) = 2x$$

$$y - f(3) = f'(3)(x - 3) \Rightarrow [y - 9 = 6 \cdot (x - 3)] \Rightarrow y = 6x - 9$$

RECTA NORMAL

$$\boxed{y - f(x_0) = \frac{-1}{f'(x_0)}(x - x_0)}$$

cont. M1)

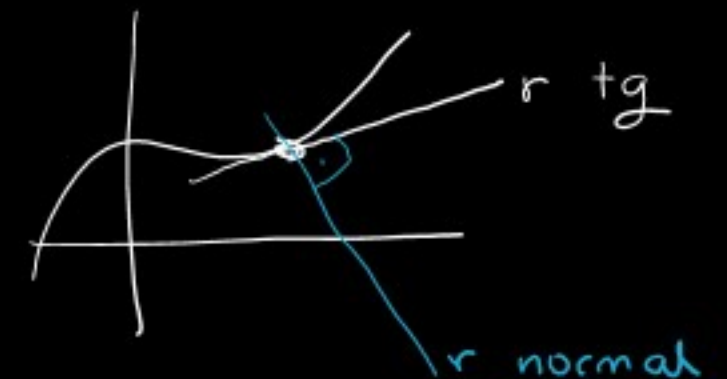
$$y - f(3) = \frac{-1}{f'(3)}(x - 3) \Rightarrow \left[y - 9 = \frac{-1}{6}(x - 3) \right]$$

OTRO EJ:

calcula recta tg

PARALELA a ...

$$\Rightarrow f'$$



- DERIVABLE $\Rightarrow$ CONTINUA (siempre)

- CONTINUA $\nRightarrow$ DERIVABLE (no lo asegura)

CONDICIÓN NECESARIA,
NO SUFICIENTE

$$\text{Derivada lat. } \underline{\text{dcha}} = f'(x_0^+) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \left[\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} \right]$$

$$\text{Derivada lat. } \underline{\text{izq}} = \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\left[f \text{ derivable en } x_0 \Leftrightarrow f'(x_0^+) = f'(x_0^-) \right] = f'(x_0) \text{ deriv. de } f \text{ en } x_0$$

Estudiar en $x = -1$ y $x = 0$ la derivabilidad de

$$f(x) = \begin{cases} -(2x+1) & \text{si } x \leq -1 \\ x^2 & \text{si } -1 < x < 0 \\ \text{sen } x & \text{si } x \geq 0 \end{cases}$$

1) Continuidad
 2) Derivabilidad

da problema? **1a1**
saltos

continua, no dan problema en el dominio

ej: $\begin{cases} x & \text{si } x < 0 \\ \frac{1}{x+2} & \text{si } x \geq 0 \end{cases}$

↑
daña problema en $x = -2$ ($\notin$ dominio)
 $\Rightarrow \text{😊}$

en $x = -1$: **CONT** $\Leftrightarrow \lim_{x \rightarrow -1^-} -(2x+1) = \lim_{x \rightarrow -1^+} x^2 = f(-1)$

$\Leftrightarrow 1 = 1 \quad \checkmark$

$f(x)$ cont. en $x = -1$

DERIV? $\Leftrightarrow \lim_{x \rightarrow -1^-} f'(x) = \lim_{x \rightarrow -1^+} f'(x)$

lo podo poner porq es derivable

$$f'(x) = \begin{cases} -2 & \text{si } x \leq -1 \\ 2x & \text{si } -1 < x < 0 \\ \cos x & \text{si } x \geq 0 \end{cases}$$

$\lim_{x \rightarrow -1^-} f'(x) = \lim_{x \rightarrow -1^-} (-2) = -2$

$\lim_{x \rightarrow -1^+} f'(x) = \lim_{x \rightarrow -1^+} 2x = -2$

$\Rightarrow \exists f'(-1) = -2$

$f(x)$ es deriv. en $x = -1$

$$f'(x) = \begin{cases} (2x-1) & \text{si } x \leq -1 \\ x^2 & -1 < x < 0 \\ \sin x & x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} -2 & \text{si } x \leq -1 \\ 2x & -1 < x < 0 \\ \cos x & x \geq 0 \end{cases}$$

no $\geq$

$$\cos\left(\frac{\pi}{2}\right) = 0$$

$$\sin\left(\frac{\pi}{2}\right) = 1$$

en $x=0$

CONT.

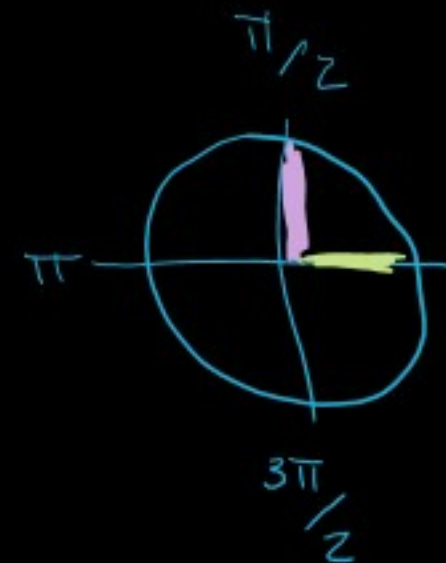
$$f(x) \text{ cont. en } x=0 \Leftrightarrow \lim_{x \rightarrow 0^-} x^2 = \lim_{x \rightarrow 0^+} \sin x = f(0)$$

" " "

0 0 0

✓

(cont.)



$$\sin(0) = 0$$

$$\cos(0) = 1$$

DERIV.

$$\lim_{x \rightarrow 0^-} (2x) = \lim_{x \rightarrow 0^+} \cos x$$

"

0

$\neq$

"

1

$\Rightarrow f(x)$ no es derivable en $x=0$

Derivada infinita:

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \infty \quad (\text{ó } -\infty)$$

(x_0 es un punto de tangente vertical)

* Derivada:

Pendiente de la recta
tg a f en un punto.

REGLAS DERIVACIÓN:

$$1) (f+g)' = f' + g'$$

$$2) (c \cdot f)' = c \cdot f'$$

$$3) (f \cdot g)' = f' \cdot g + f \cdot g'$$

$$4) \left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

$$5) (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

REGLA DE LA CADENA:

$$\text{Si } h(x) = g(f(x)) \Rightarrow$$

$$\Rightarrow h'(x) = g'(f(x)) \cdot \underline{f'(x)}$$

Por la derivada
de lo de dentro

M3

$$f(x) = \log\left(\frac{x^2}{3-x}\right)$$

$$f'(x) = \frac{1}{\left(\frac{x^2}{3-x}\right)} \cdot \left(\frac{2x \cdot (3-x) - x^2 \cdot (-1)}{(3-x)^2} \right) =$$

$$= \frac{\cancel{3-x}}{x^2} \cdot \frac{6x - 2x^2 + x^2}{(3-x)^{\cancel{2}}} = \frac{6x - x^2}{x^2 \cdot (3-x)} = \frac{\cancel{x}(6-x)}{\cancel{x}(3-x)} = \boxed{\frac{6-x}{3-x} = f'(x)}$$

$$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$$

$$y = \ln(f(x)) \Rightarrow y' = \frac{1}{f(x)} \cdot f'(x)$$

M4

Recta tg y normal a $f(x) = x^{\text{sen } x}$ en $x_0 = \frac{\pi}{2}$ $f = x$
 $g = \text{sen } x$

$$y = f^g \Rightarrow y' = y \cdot (g' \ln f + g \cdot f^{g-1} \cdot f') \Rightarrow y' = y \cdot (g' \ln f + g \cdot \frac{1}{f} \cdot f')$$

$$f'(x) = x^{\text{sen } x} \cdot (\cos x \cdot \ln x + \text{sen } x \cdot x^{\text{sen } x - 1} \cdot 1)$$

recta tg: $y - f(x_0) = f'(x_0)(x - x_0)$

$$x_0 = \frac{\pi}{2} \quad y - f\left(\frac{\pi}{2}\right) = f'\left(\frac{\pi}{2}\right) \cdot \left(x - \frac{\pi}{2}\right)$$

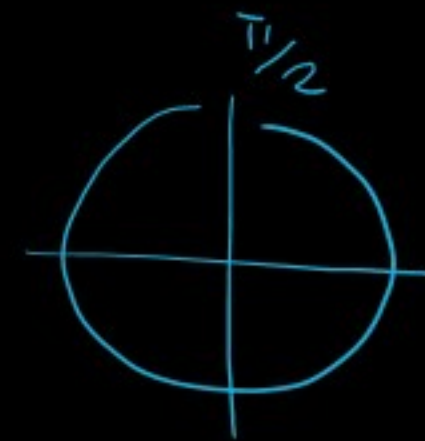
$$f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^{\text{sen } \frac{\pi}{2}} = \frac{\pi}{2}$$

$$y - \frac{\pi}{2} = \frac{\pi}{2} \left(x - \frac{\pi}{2}\right)$$

$$f'\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right)^1 \cdot \left(0 + 1 \cdot \frac{\pi}{2}^0 \cdot 1\right) = \frac{\pi}{2}$$

recta normal: $y - f(x_0) = \frac{-1}{f'(x_0)} (x - x_0)$

$$y - \frac{\pi}{2} = \frac{-1}{\frac{\pi}{2}} \left(x - \frac{\pi}{2}\right) \Rightarrow \left[y - \frac{\pi}{2} = -\frac{2}{\pi} \left(x - \frac{\pi}{2}\right)\right]$$



$$\text{sen}\left(\frac{\pi}{2}\right) = 1$$

$$\cos\left(\frac{\pi}{2}\right) = 0$$

APROX. POR RECTA TG:



si $f'(x_0) \neq 0$ y x está "cerca" de $x_0 \Rightarrow$

$$f(x) \approx \text{recta tg}$$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$y = f(x_0) + f'(x_0)(x - x_0)$$

ej M4. Valor aproximado de $\sqrt[4]{1.06}$

$$f(x) = \sqrt[4]{x} = x^{1/4} \Rightarrow f'(x) = \frac{1}{4} x^{-3/4} \quad \left[f(x) \approx f(x_0) + f'(x_0)(x - x_0) \right]$$

$x = 1.06$

x_0 CERCA DE x (y fácil) $\Rightarrow x_0 = 1$

$$\sqrt[4]{x} \approx \sqrt[4]{1} + \frac{1}{4} \cdot (1)^{-3/4} \cdot (x - 1) \quad (x = 1.06)$$

$$\sqrt[4]{1.06} \approx 1 + \frac{1}{4} \cdot (1.06 - 1)$$

$$\sqrt[4]{1.06} \approx 1 + \frac{1}{4} \cdot 0.06 \Rightarrow \boxed{\sqrt[4]{1.06} \approx 1.015}$$

EJ (P1)

$e^{0.1}$ aprox?

PASOS:

1) Elijo función y punto x_0

$$f(x) = e^x \quad x = 0.1 \quad x_0 = 0$$

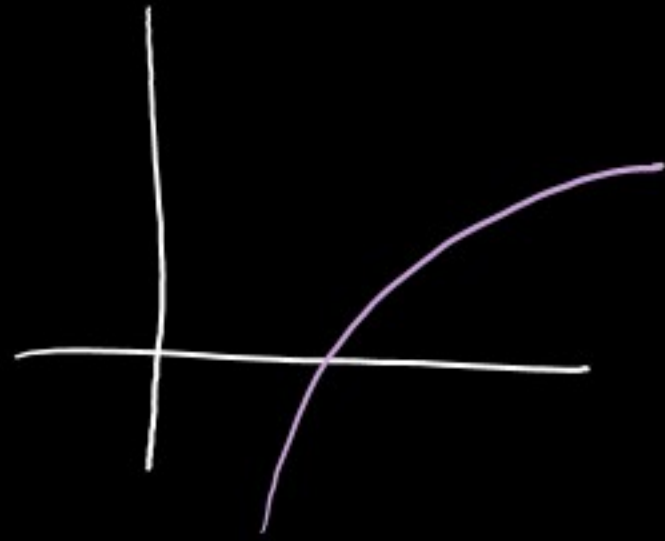
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

2) Aprox. por la tangente

$$e^x \approx e^0 + e^0(x - 0) \Rightarrow e^x \approx 1 + x$$

3) Sustituyo en mi punto

$$e^{0.1} \approx 1 + 0.1 \Rightarrow \boxed{e^{0.1} \approx 1.01}$$



f cresce

$$f' > 0$$

$$f'' < 0$$

Exemplo 2ª derivada: $f''(x)?$ $f(x) = \frac{1}{\sqrt{1+x}} + \sin^2 x = (1+x)^{-1/2} + (\sin x)^2$

$$f'(x) = -\frac{1}{2} (1+x)^{-3/2} + 2(\sin x) \cdot \cos x$$

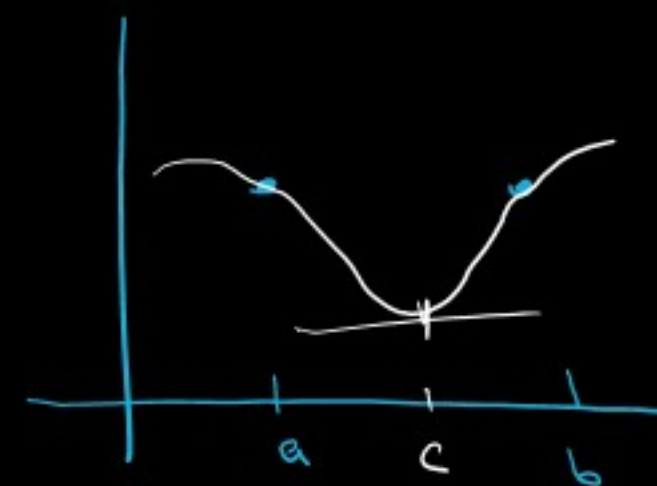
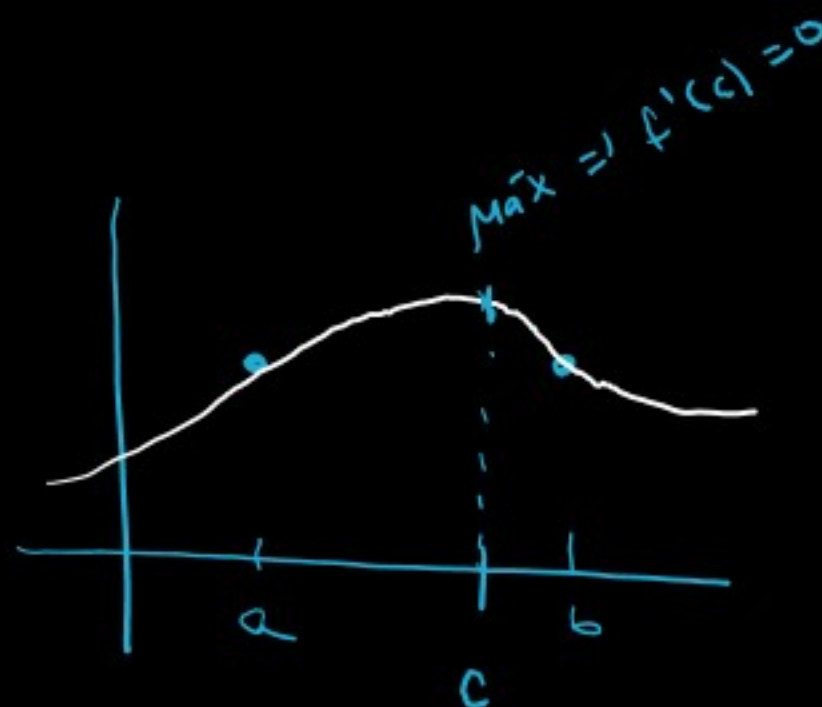
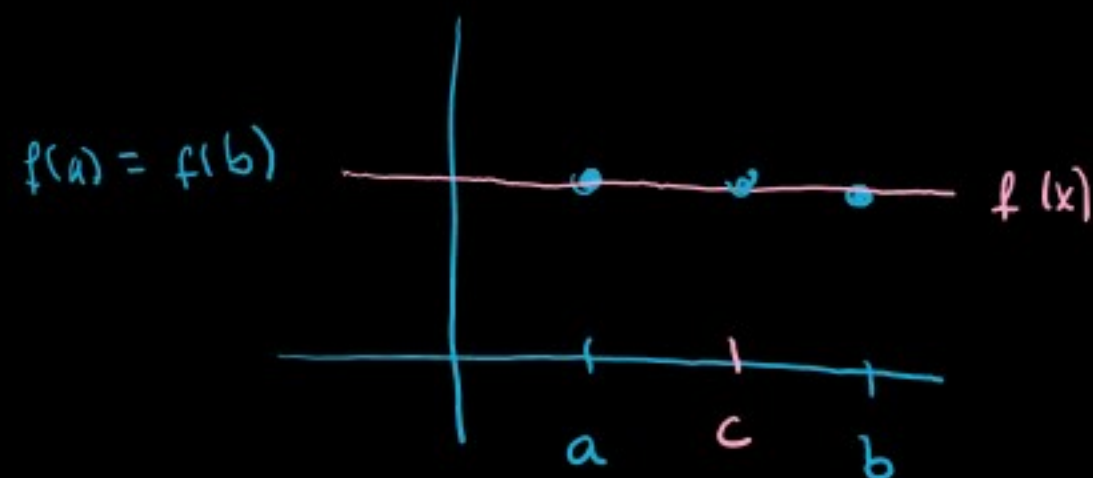
$$f''(x) = -\frac{1}{2} \cdot -\frac{3}{2} (1+x)^{-5/2} + 2 \cdot [\cos x \cdot \cos x + \sin x \cdot (-\sin x)]$$

$$= \left[\frac{3}{4} (1+x)^{-5/2} + 2 \cdot (\cos^2 x - \sin^2 x) \right]$$

TEOREMA DE ROLLE

Sea f continua en $[a, b]$ y derivable en (a, b) tq $f(a) = f(b) \Rightarrow$

$$\Rightarrow \exists c \in (a, b), f'(c) = 0$$



EJ. (P)

$$f(x) = \sin[(x-1)(x-2)^2(x-3)(x-4)^2] + \cos[(x-1)^2(x-2)(x-3)^3(x-4)] - 1$$

Demstrar que $\exists x_0, f'''(x_0) = 0$

Raíces de f ? $f(\text{raíz}) = 0$

$x=1, x=2, x=3, x=4 \rightarrow$ están a la misma altura

$$(f(1) = f(2) = f(3) = f(4) = 0)$$



$$\exists f'(c) = 0$$

$\exists$ raíz de f'

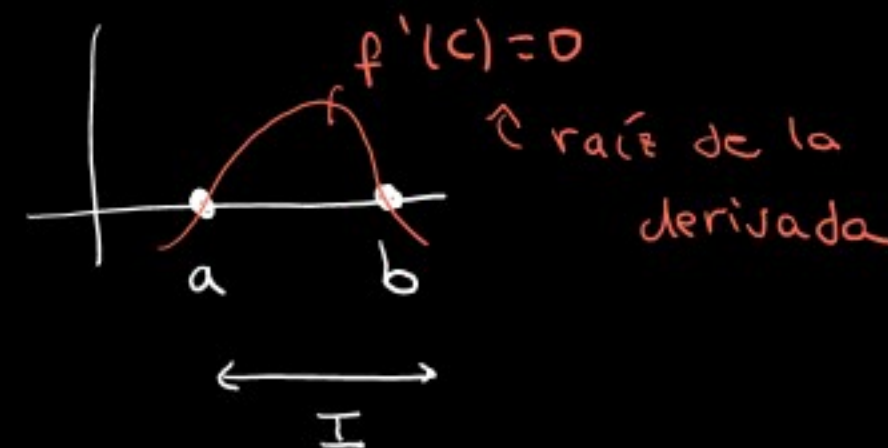
(3 raíces de f')



$$f'(r_1) = 0 = f'(r_2) = f'(r_3) \quad (\text{SON RAÍCES})$$

TH. ROLLE

Entre 2 raíces de f en un intervalo I , $\exists$ al menos una raíz de su derivada



$\Rightarrow f''(x)$ tiene al menos dos raíces $\Rightarrow \exists c_1, c_2, f''(c_1) = f''(c_2) = 0$



$\Rightarrow \exists$ al menos una raíz de $f'''(x) \neq$

REPASO T1:

DUDA:

$$f(x) = (\sqrt[3]{x-1})^2 \cdot \sin(x^2-1) \cdot \operatorname{tg} x$$

orden y PP $x_0 = 1$

$$\underbrace{\sin(f(x))}_{\rightarrow 0} \approx f(x)$$

TEORÍA: $x_0 \neq \infty$

$$\lim_{x \rightarrow x_0} \frac{f(x)}{(x-x_0)^p} = A \neq 0, \infty$$

$$PP(f) = A(x-x_0)^p$$

$$\lim_{x \rightarrow 1} \frac{(\sqrt[3]{x-1})^2 \cdot \sin(\underbrace{x^2-1}_{\rightarrow 0}) \cdot \operatorname{tg}(\underbrace{x}_{\rightarrow 1})}{(x-1)^p} =$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)^{2/3} \cdot (x^2-1) \cdot \operatorname{tg} x}{(x-1)^p} = \lim_{x \rightarrow 1} \frac{(x-1)^{2/3} \cdot (x-1)(x+1) \cdot \operatorname{tg} x}{(x-1)^p} =$$

$$= \lim_{x \rightarrow 1} \frac{(x-1)^{\frac{5}{3}} \cdot (x+1) \operatorname{tg} x}{(x-1)^p}$$

$$p = \frac{5}{3}$$

$$\lim_{x \rightarrow 1} \frac{(x-1)^{\cancel{5/3}} \cdot (x+1) \operatorname{tg} x}{(x-1)^{5/3}} = \boxed{2 \cdot \operatorname{tg} 1 = A}$$

$$PP(f) = 2 \operatorname{tg} 1 \cdot (x-1)^{5/3}$$

EJERCICIO:

$$f(x) = (1 - \cos(3x)) + \operatorname{tg}(x) + \log(1 + \sqrt[3]{x}) \quad \text{en } x_0 = 0 \quad \text{Orden? PP?}$$

$$\lim_{x \rightarrow 0} \frac{(1 - \cos(3x)) + \operatorname{tg}(x) + \log(1 + \sqrt[3]{x})}{(x-0)^p} = \lim_{x \rightarrow 0} \frac{\frac{9x^2}{2} + x + \sqrt[3]{x}}{x^p} =$$

$$= \lim_{x \rightarrow 0} \left[\underbrace{\frac{9/2 x^2}{x^p}} + \underbrace{\frac{x}{x^p}} + \underbrace{\frac{x^{1/3}}{x^p}} \right] = A \Leftrightarrow \boxed{p = \frac{1}{3}} \quad \boxed{A = 1} \quad \text{PP}(f) = A(x-x_0)^p = 1 \cdot (x-0)^{1/3} = \boxed{x^{1/3}}$$

PP(f)

Probamos

$$p=2: \lim_{x \rightarrow 0} \left(\underbrace{\frac{9/2 x^2}{x^2}}_{\checkmark} + \underbrace{\frac{x}{x^2}}_{\text{va a tender a 0 m\u00e1s r\u00e1pido q el numerador} \neq \infty} + \frac{x^{1/3}}{x^2} \right)$$

$$p=1: \lim_{x \rightarrow 0} \left(\underbrace{\frac{9/2 x^2}{x}}_{0} + \underbrace{\frac{x}{x}}_{\checkmark} + \underbrace{\frac{x^{1/3}}{x}}_{\text{tiende a 0 m\u00e1s r\u00e1pido} \Rightarrow \infty} \right)$$

$$p=\frac{1}{3}: \lim_{x \rightarrow 0} \left(\underbrace{\frac{9/2 x^2}{x^{1/3}}}_{0} + \underbrace{\frac{x}{x^{1/3}}}_{0} + \underbrace{\frac{x^{1/3}}{x^{1/3}}}_{1} \right)$$

ejemplo:

$$\lim_{x \rightarrow 0} \frac{\arcsen(\cancel{x^2}) \cdot \cos(x) \cdot \text{tg}(\cancel{\sqrt[3]{x}}) \cdot \log(1 + \cancel{\sqrt[3]{x^2}})}{x^3 + \text{sen}(\cancel{x^3})} =$$

$$= \lim_{x \rightarrow 0} \frac{x^2 \cdot \cos x \cdot \overbrace{\sqrt[3]{x} \cdot \sqrt[3]{x^2}}^x}{x^3 + x^5} = \lim_{x \rightarrow 0} \frac{x^2 \cdot \cos x \cdot x}{x^3 (1 + x^2)} = \lim_{x \rightarrow 0} \frac{x^3 \cdot \cos x}{x^3 \cdot (1 + x^2)} =$$

$$= \lim_{x \rightarrow 0} \frac{\cos x}{1 + x^2} = \frac{1}{1} = \boxed{1}$$

$$\left[\begin{aligned} 1 - \cos f(x) &\approx \frac{[f(x)]^2}{2} \Rightarrow \\ \Rightarrow 1 - \frac{[f(x)]^2}{2} &\approx \cos(f(x)) \end{aligned} \right]$$

EJERCIO INTER 2023:

Hallar A y B tal que

es cont. en $x=5$

deriv. en $x=0$

$$f(x) = \begin{cases} \frac{\ln(5 + \sin^5 x - 4 \cos(x^3)) (4 + (\arctg^2 x) \ln(-x))}{(1 - \cos^4 x)(e^{x^2} - 1)} & \text{si } x < 0 \\ (A(5-x) + B) \cdot x & \text{si } 0 \leq x \leq 5 \\ \frac{50 \cdot (\sqrt[3]{x+22} - 3) \cdot \ln\left(\frac{x^3}{125}\right) (e^{-\sqrt[3]{(x-5)^2}} - 1)}{\sec^2\left(\frac{x-5}{3}\right) \cos(x-5)} & \text{si } x > 5 \end{cases}$$

Sol:

$$B = -2$$

$$A = \frac{4}{5}$$